

ANISOTROPIC GENERALIZATIONS OF DE SITTER SPACETIME

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ABSTRACT. It is known that de Sitter spacetime can be seen as the solution of field equation for completely isotropic matter. In the present paper a new class of exact solutions in spherical symmetry is found and discussed, such that the energy-momentum tensor has two 2-dimensional distinct isotropic subspaces.

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1. INTRODUCTION

As is well known, the energy-momentum tensor describing ordinary matter is of the Plebański type $[T - S_1 - S_2 - S_3]_{(1111)}$, i.e. it admits one timelike and three spacelike eigenvectors (u^μ and $(m^a)^\mu$ with $a = 1, 2, 3$, say). The eigenvalue corresponding to the timelike vector field u is (minus) the energy density of the matter ϵ , while the three spacelike eigenvectors are the principal stresses λ_a ($a = 1, 2, 3$) (obviously, the perfect fluid is space-isotropic, so that for a fluid the spacelike eigenvalues coincide). Using as tetrad basis that defined by the eigenvectors, the energy momentum tensor can be written in the form

$$(1.1) \quad T_\nu^\mu = \epsilon u^\mu u_\nu + \sum_a \lambda_a (m^a)^\mu (m^a)_\nu$$

From now on we work in spherical symmetry (see [6]), where only two spacelike eigenvalues can be distinct, so that the canonical form is

$$(1.2) \quad T_\nu^\mu = \epsilon u^\mu u_\nu + p_r m^\mu m_\nu + p_t \Delta_\nu^\mu,$$

where m^μ denotes the unit radial vector and Δ denotes the projector onto the two-dimensional spacelike subspace orthogonal to m . The radial and tangential stresses are respectively denoted by p_r and p_t .

The de Sitter spacetime can be viewed as a vacuum solution with nonzero "lambda" term but also, as is well known, as the spacetime originated by a lambda term "source", i.e. as the solution of the Einstein field equation in matter having $T_\nu^\mu = -\epsilon_0 \delta_\nu^\mu$ where ϵ_0 is now constant due to the field equations for matter, which obviously imply $\partial_\nu \epsilon_0 = 0$. In the formula (1.2), one thus has $\epsilon = -p_t = -p_r = \epsilon_0$ so that the energy tensor is completely degenerate.

One may now ask if there are ways, in which it is possible to weaken the hypothesis of complete degeneracy in a "minimal" way, i.e. retaining two degenerate subspaces, thereby obtaining "anisotropic generalizations" of de Sitter spacetime. It is obvious, that perfect fluids cannot achieve this goal. However, if the stress is anisotropic then one can search for solutions described by

$$(1.3) \quad T_\nu^\mu = -A \Gamma_\nu^\mu + p_t \Delta_\nu^\mu$$

where A is some function and Γ is the unit tensor living in the two-dimensional subspace spanned by the timelike and the radial spacelike eigenvectors (i.e. $\Gamma_\nu^\mu = -u^\mu u_\nu + m_\mu m^\mu$). Interestingly enough, the equations of motion for the matter now shows that ϵ is a function depending on R only, where R is the comoving area radius.

In the present paper this class of solutions is derived, together with minimal physical requirements that have to be imposed. It also includes models used to build up regular black hole interiors [2, 3, 8, 9], where a regular solution is used to replace a singular core (e.g. Schwarzschild). It may be worthwhile noticing that here the starting point is given by a condition on the constitutive equation (2.4), (see (2.7) below) recovering a solution which in principle is neither static nor regular.

The equation of state for a general material in spherical symmetry can be given in terms of a *state function* (see e.g. [7])

$$(2.4) \quad \epsilon = \epsilon(r, R, \eta),$$

where $\eta = e^{-2\lambda}$, in such a way that the stresses, which are in general anisotropic, are given by the following relations:

$$(2.5) \quad p_r = -\epsilon + 2\eta \frac{\partial \epsilon(r, R, \eta)}{\partial \eta}, \quad p_t = -\epsilon - \frac{R}{2} \frac{\partial \epsilon(r, R, \eta)}{\partial R}.$$

It follows, that since, from (1.3), the solutions we are searching for are uniquely characterized by the condition that

$$(2.6) \quad \epsilon + p_r = 0,$$

then ϵ has to be independent from η :

$$(2.7) \quad \frac{\partial \epsilon}{\partial \eta} = 0.$$

Substituting (2.5) in (2.2c) and using $\epsilon'(r, t) = \frac{\partial \epsilon(r, R)}{\partial r} + \frac{\partial \epsilon(r, R)}{\partial R} R'$ coming from (2.7), we get also $\frac{\partial \epsilon}{\partial r} = 0$, that is ϵ given by (2.4) must be a function of R only. We are thus left with the following energy-momentum tensor:

$$(2.8) \quad T_\nu^\mu = -\epsilon \Gamma_\nu^\mu - \left(\epsilon + \frac{R}{2} \frac{d\epsilon}{dR} \right) \Delta_\nu^\mu,$$

and since equations (2.2a) now read

$$m' = 4\pi \epsilon(R) R^2 R', \quad \dot{m} = 4\pi \epsilon(R) R^2 \dot{R},$$

it must be

$$(2.9) \quad m(R) = 4\pi \int_0^R \epsilon(\sigma) \sigma^2 d\sigma + m_0.$$

In order for this solution to satisfy minimal requirements of acceptability, the *weak energy condition* (w.e.c.) is imposed, that reads

$$(2.10) \quad \epsilon \geq 0, \quad \epsilon + p_r \geq 0, \quad \epsilon + p_t \geq 0$$

(a basic reference for a discussion of energy conditions is [5]). In this case (2.5) implies that w.e.c. is satisfied if $\epsilon(R)$ is a non negative and not increasing function of R .

We are left with the system (2.2b), (2.3), and (2.9) in the unknown R , λ , ν , ϵ and m . In principle (2.2b) should be integrated in order to obtain exact solutions, but recalling, from (2.3) and (2.9), that $(R'e^{-\lambda})^2 - (\dot{R}e^{-\nu})^2$ is a function of R only, we will limit ourselves to the case when the two addenda are separately function of R only:

$$(2.11) \quad R'e^{-\lambda} \equiv a(R), \quad \dot{R}e^{-\nu} \equiv b(R),$$

where a and b are two *prescribed* functions of R .

Using condition (2.2b) we find, up to time reparameterization, the following expressions:

$$\lambda = \log(f(r)b(R)), \quad \nu = \log a(R),$$

$$m(R) = \frac{R}{2}[1 - a^2(R) + b^2(R)], \quad \epsilon(R) = \frac{1}{4\pi R^2} \frac{dm(R)}{dR},$$

and the two (compatible) equations for $R(r, t)$:

$$(2.12) \quad \dot{R} = a(R)b(R), \quad R' = f(r)a(R)b(R).$$

Here the function $f(r)$ arises as an integration term, and since the curve $R(r, 0)$ is the initial data that will be conveniently taken equal to r , it is $R'(r, 0) = 1$ and then $f(r) = (a(r)b(r))^{-1}$.

With the position

$$v(R) := a(R)b(R),$$

equations (2.12) reads

$$(2.13) \quad \dot{R} = v(R), \quad R' = \frac{v(R)}{v(r)},$$

that integrate to give

$$(2.14) \quad R(r, t) = V^{-1}(t + V(r)),$$

where $V(\sigma)$ is a primitive of $\frac{1}{v(\sigma)}$ (note that V is invertible). The line element then takes the form

$$(2.15) \quad ds^2 = -a^2(R)dt^2 + \frac{v^2(R)}{v^2(r)a^2(R)}dr^2 + R^2d\Omega^2,$$

with $R = R(r, t)$ given by (2.14).

3. PHYSICAL INTERPRETATION AND REMARKS

Defining the function

$$(3.1) \quad \chi(R) = 1 - \frac{2m(R)}{R} = a^2(R) - b^2(R) = a^2(R) - \frac{v^2(R)}{a^2(R)},$$

it can be seen that two suitable coordinate changes map the regions $\{\chi > 0\}$ and $\{\chi < 0\}$ respectively into a *variation of mass* of Schwarzschild solution. The sign of χ defines the character of R viewed as a coordinate: if χ is positive, R may be regarded as a "length", if χ is negative R can be seen as a "time" instead.

It can be noticed that these solutions can arise from Kerr-Schild geometry, since another suitable coordinate change may be applied, to recover the form

$$(3.2) \quad ds^2 = (-d\bar{t}^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2) + (1 - \chi(\bar{r}))(d\bar{t} + d\bar{r})^2,$$

that shows, by the way, that $\chi = 0$ is a removable singularity. The family outlined here in fact contains Minkowski, Schwarzschild and de Sitter spacetimes as particular cases, corresponding to choosing the function $m(R)$ respectively equal to 0, to m_0 (constant), and to $\frac{4}{3}\epsilon_0\pi R^3$.

As sketched before, the line element of these metrics are formally analogue to Schwarzschild exterior and black hole (coinciding with them if $m(R)$ is constant), so

it is not surprising that $\chi = 0$ is not a true singularity. Computation of Kretschmann scalar $K = R_{abcd}R^{abcd}$ yields

$$(3.3) \quad K = 4 \frac{(\chi - 1)^2}{R^4} + 4 \frac{(\chi')^2}{R^2} + (\chi'')^2,$$

and this suggest that the only singularity can occur at $R = 0$. Let us assume m (and therefore χ) analytic in a right neighborhood of 0, and let $\alpha \geq 0$ such that $m(R) \cong R^\alpha$. It is easily seen that metric regularity at $R = 0$ occurs if and only if $\alpha \geq 3$. On the other side, since w.e.c. (2.10) in terms of m reads

$$(3.4) \quad m'(R) \geq 0, \quad m''(R) - \frac{2}{R}m'(R) \leq 0,$$

in order for (3.4) to be satisfied near $R = 0$ it must be $\alpha \leq 3$. Therefore, the only case for a physically acceptable metric not to be singular at $R = 0$ is when it behaves asymptotically as de Sitter spacetime as R approaches 0 (that is, $m(R) \cong R^3$). In all other cases (allowed by w.e.c.) $R = 0$ is a true singularity.

The solutions here found can be matched to Schwarzschild spacetime if and only if ϵ vanishes on a $R = \text{const.}$ surface, that is there exists $R_b > 0$ such that $\frac{dm}{dR}(R_b) = 0$. This can be performed either in the static region (i.e. where $\chi > 0$), or in the non static one. In the first case, we obtain a globally static object.

If, instead, we allow χ to vanish for some $R_0 > 0$ (Cauchy horizon), the solution may enter the non static region, where the matching can be performed obtaining physically valid black hole interior models. This is not in contradiction with the result proved by Baumgarte and Rendall in [1], stating that the inequality $\chi(R) > 0$ remains true until radial pressure vanishes. Indeed, the hypothesis $\epsilon + p_r > 0$ is crucial for the argument used in [1], whereas in our case the w.e.c. is satisfied at its borderline, that is $\epsilon + p_r \equiv 0$. As pointed out before, these solutions are regular only if they have a de Sitter-like behavior as $R \rightarrow 0^+$, other regular choices being forbidden by w.e.c.. The junction can even be done between a true de Sitter core and a Schwarzschild spacetime, but this time an intermediate thick shell is needed in the middle, to ensure continuity of radial pressure. An example is given by

$$(3.5a) \quad m(R) = \begin{cases} \frac{4}{3}\pi\epsilon_0 R^3, & R \in [0, kM[\\ \frac{4}{3}\pi\epsilon_0(R - 2kM)^3 + 2M, & R \in [kM, 2kM[\\ 2M, & R \in [2kM, +\infty) \end{cases}$$

where k is a constant such that, to ensure continuity, the Schwarzschild mass M equals twice the mass of de Sitter $\frac{4}{3}\pi\epsilon_0(kM)^3$. This choice yields the following energy density function:

$$(3.5b) \quad \epsilon(R) = \begin{cases} \epsilon_0, & R \in [0, kM[\\ \epsilon_0 \left(1 - \frac{2kM}{R}\right)^2, & R \in [kM, 2kM[\\ 0, & R \in [2kM, +\infty). \end{cases}$$

The limitation $k \leq 1$ gives a relation between the parameters of de Sitter and Schwarzschild solutions (i.e. $\epsilon_0 M^2 \leq \frac{3}{8\pi}$) to be satisfied in order to obtain a regular black hole interior model with a single Cauchy horizon.

Regular Schwarzschild black holes have been studied by several authors [2, 3, 8, 9], and explicit solutions have been build up, with a number of Cauchy horizons even greater than one before energy density vanishes (see [9] for an example).

Of course, matching with Schwarzschild spacetime may be also performed if $m(R) \cong R^\alpha$ with $\alpha < 3$, this operation resulting in replacing Schwarzschild black hole with a non flat core still possessing a singularity at $R = 0$. As an example, if $m(R)$ is a second order polynomial in R ,

$$(3.6) \quad m(R) = -kR^2 + 4kMR, \quad R \in [0, 2M],$$

the junction is done at $R = 2M$, and the condition $k \geq \frac{1}{4M}$ ensures that this matching is made in the non static region. Therefore, in this case there are no Cauchy horizons.

We also notice that the above constructions are made supposing continuity of both the metric and radial pressure p_r . For instance, if the core is a Schwarzschild solution, we cannot perform a matching with one of our solution at $R_0 > 0$, since continuity of p_r is lost at R_0 . Anyway, if discontinuities of p_r at junctions are allowed, other spacetimes can be built. Indeed, matching between two variations of mass of Schwarzschild spacetime – generated by an "inside-shell" mass $m(R)$ and an "outside-shell" mass $M(R)$ – can be performed at R_0 such that, for instance, both $\chi_m(R_0)$ and $\chi_M(R_0)$ are positive. In this case it must be checked that the surface stress-energy tensor (i.e. related to the surface $\mathcal{S} = \{R = R_0\}$) obeys weak energy condition. Following the method in [10] the surface energy density and pressure are found to be

$$(3.7a) \quad \epsilon_S = \frac{1}{4\pi R_0} \left(\sqrt{\chi_m(R_0)} - \sqrt{\chi_M(R_0)} \right),$$

$$(3.7b) \quad p_S = \frac{1}{8\pi R_0^2} \left[\frac{(1 - M'(R_0))R_0 - M(R_0)}{\sqrt{\chi_M(R_0)}} - \frac{(1 - m'(R_0))R_0 - m(R_0)}{\sqrt{\chi_m(R_0)}} \right],$$

and for the solution to be physically acceptable it must be $\epsilon_S \geq 0$ (that is, $M(R_0) \geq m(R_0)$) and $\epsilon_S + p_S \geq 0$. A particular case of a Schwarzschild region surrounded by a massive shell is studied in [4], where the region outside shell can also be viewed as a Schwarzschild solution, under a suitable coordinate change.

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